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# NECESSARY CONDITIONS FOR SIMILAR SOLUTIONS OF PROBLEMS OF TURBULENT-GAS DYNAMICS

Barbara K. Crowley

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CONTENTS

# NECESSARY CONDITIONS FOR SIMILAR SOLUTIONS OF PROBLEMS OF TURBULENT-GAS DYNAMICS

## ABSTRACT

The quasi-one-dimensional conservation equations with friction, heat, and mass sources are considered. Necessary conditions for similar solutions of these equations are mathematically derived, and these conditions are verified by sample computer calculations.

## INTRODUCTION

PUFL is a quasi-one-dimensional, "almost-Lagrangian" computer code for solving the partial differential equations of mass, momentum, and energy conservation for gas flows in pipes.[1] The differential equations are cast in the Lagrangian form, but mass sources are allowed. Also, frictional forces and energy sources or sinks are considered.

In addition to providing a better understanding of the equations, there are two major reasons for seeking similar solutions of the PUFL equations. First, the number of calculations that need to be considered could be reduced; and, secondly, knowledge of the mathematical conditions for similarity could provide a basis for designing scaled experiments.

In Section I of this report, the general conservation equations for pipe flows with ablation are given and briefly discussed. The conditions necessary for similar solutions are then derived. The auxiliary equations used by the PUFL calculation to simulate ablation are presented in Section II. The necessary conditions that the similarity conditions from Section I impose on the auxiliary equations and on the physical parameters are discussed. Section III presents some sample PUFL calculations which indicate that by using the derived conditions, similar solutions are obtained.

## SECTION I:
## DERIVATION OF SIMILAR CONDITIONS FOR THE PUFL EQUATIONS

The conservation equations that are the basis of the PUFL model are presented below. The detailed derivation of these equations may be found in Ref. 1.

### Continuity

The continuity equation is

$$\frac{Dm}{Dt} = \frac{D}{Dt}(\rho V) = S\dot{m},$$

where $\dot{m}$ is the mass flux (mass/area-time) entering a volume homogeneously along the side walls of the pipe that has a surface area S. The entering mass is assumed to mix instantaneously with the material already present.

For extensive (mass-dependent) variables such as the volume, the substantial derivative, D/Dt, is simply d/dt. Hence, after rearranging, the above equation becomes

$$\frac{D\rho}{Dt} = -\frac{\rho}{V} \cdot \frac{dV}{dt} + \frac{S\dot{m}}{V} .$$

### Momentum

The equation for the conservation of momentum is

$$\frac{D}{Dt}(m\underset{\sim}{u}) = \Sigma\underset{\sim}{F},$$

where $\Sigma\underset{\sim}{F}$ is the vector sum of the forces acting on an element. Since the mass of a particle with mass sources varys with respect to time, the mass is inside the derivative operator.

The one-dimensional momentum equation used by PUFL is

$$\frac{Du}{Dt} = \frac{1}{m}\left[\dot{m}S(u_W - u) - V\frac{\partial p}{\partial x} - \tau_w S\right] .$$

The first term represents the adjustment of momentum in the gas due to the mass flux ($\dot{m}$). The velocity of entering material is $u_w$ in an x direction ($u_w$ may be positive or negative). The accelerating force due to pressure differentials is denoted by $-V\frac{\partial p}{\partial x}$. The term $\tau_w S$ is the retarding frictional force exerted on the gas flow by a pipe wall having a surface area S. Viscous forces within the gas are assumed to be zero. Also, body forces are assumed to be negligible.

## Energy

The conservation of total energy – kinetic plus internal – is considered by PUFL. By using the momentum equation for velocities, the total energy equation reduces to an equation for specific internal energy:

$$\frac{De}{Dt} = \frac{1}{m}\left\{\dot{m}S\left[\frac{(u - u_w)^2}{2} + (e_w - e)\right] - p\frac{dV}{dt} + \tau_w S\left|u\right| + \dot{H}\right\}.$$

The first term represents the modification of specific internal energy within a gas by the total energy of the entering mass. The expression $p\frac{dV}{dt}$ is the usual work term, and $\tau_w S\left|u\right|$ expresses the conversion of the translational kinetic energy of a gas into internal energy of heat due to the shear stress ($\tau_w$) at the wall. The rate of energy change within a gas due to sources or sinks is denoted by $\dot{H}$. This term may be related to the mass entering the gas by using an additional ablation equation.

## Derivation of the Similar Conditions

In order to obtain relationships necessary for similar solutions, the conservation equations just described are now examined. The approach used in the following discussion is equivalent to the familiar approach frequently used in dimensional analysis to nondimensionalize flows by the use of characteristic or reference parameters. See Appendix A for verification that this approach is equivalent.

A dimensional-analysis approach, using the Vaschy-Buckingham $\pi$ theorem,[2] also yields the same results as those derived here. Although the approach used here is more laborious than the $\pi$ method, it tends to impart a feeling of being more straightforward when applied to a number of relatively unfamiliar equations and parameters.

Consider two gas flows, both of which can be simulated by the PUFL equations. Let one flow be described by a set of unprimed variables, the other by a set of primed variables. The flows are examined at two instances when comparison of the two flows is to start. At these instances, ratios are obtained for all of the pertinent variables

and parameters in the PUFL equations. These dimensionless ratios, which relate one flow to the other, are denoted by subscripts and are defined as follows:

| | | | |
|---|---|---|---|
| Axial distance: | $X_0 = \frac{x}{x'}$ | Radial dimensional: | $R_0 = \frac{r}{r'}$ |
| Density: | $\rho_0 = \frac{\rho}{\rho'}$ | Pressure: | $p_0 = \frac{p}{p'}$ |
| Velocity: | $U_0 = \frac{u}{u'}$ | Specific internal energy: | $e_0 = \frac{e}{e'}$ |
| Volume: | $V_0 = \frac{V}{V'}$ | Surface area: | $S_0 = \frac{S}{S'}$ |
| Mass: | $M_0 = \frac{m}{m'}$ | Wall stress: | $\tau_{w0} = \frac{\tau_w}{\tau_w'}$ |
| Rate of energy deposition: | $H_0 = \frac{\dot{H}}{\dot{H}'}$ | Dimensionless coefficient of friction: | $C_{f0} = \frac{C_f}{C_f'}$ |
| Flux of mass entering the flow: | $M_0 = \frac{\dot{m}}{\dot{m}'}$ | Specific internal energy of entering mass: | $e_{w0} = \frac{e_w}{e_w'}$ |
| Time intervals after the instant at which comparison starts: | $t_0 = \frac{t}{t'}$ | Specific energy of vaporization: | $E_{v0} = \frac{E_v}{E_v'}$ |
| Dimensionless coefficient of heat transfer: | $C_{H0} = \frac{C_H}{C_H'}$ | Turbulent-transpiration coefficient: | $\eta_0 = \frac{\eta}{\eta'}$ |
| Ratio of specific heats: | $\gamma_0 = \frac{\gamma}{\gamma'}$ | Velocity of entering mass: | $U_{w0} = \frac{u_w}{u_w'}$ |

Derivatives for an arbitrary dependent variable ($\phi$) in the two flows are related by

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial x'} \cdot \frac{\partial x'}{\partial x} = \frac{\partial x'}{\partial x} \cdot \frac{\partial\phi}{\partial x'} = \frac{1}{X_0} \cdot \frac{\partial\phi}{\partial x'}$$

and by

$$\frac{D\phi}{Dt} = \frac{D\phi}{Dt'} \cdot \frac{Dt'}{Dt} = \frac{Dt'}{Dt} \cdot \frac{D\phi}{Dt'} = \frac{1}{t_0} \cdot \frac{D\phi}{Dt'} .$$

The above relationships are now used in the conservation equations. For continuity, this means that

$$\frac{D\rho}{Dt} = -\frac{\rho}{V} \cdot \frac{dV}{dt} + \frac{S\dot{m}}{V} ,$$

$$\frac{\rho_0}{t_0} \cdot \frac{D\rho'}{Dt'} = -\frac{\rho_0 \rho'}{V_0 V'} \cdot \frac{V0}{t_0} \cdot \frac{dV'}{dt'} + \frac{S_0 S' \dot{m}' \dot{M}_0}{V_0 V'} ,$$

and

$$\frac{D\rho'}{Dt'} = -\frac{\rho'}{V'} \cdot \frac{dV'}{dt'} + \frac{S_0 \dot{M}_0 t_0}{\rho_0 V_0} \cdot \frac{S' \dot{m}'}{V'} .$$

Hence, if $\frac{S_0 \dot{M}_0 t_0}{\rho_0 V_0} = 1$, the continuity equations for two physical systems are similar.

One interpretation of $S_0 \dot{M}_0 t_0 = \rho_0 V_0$ is as follows. Suppose that two flows are observed over time intervals t and t' respectively. If the ratio of the change in mass to the original (or final) mass is the same for both flows, then $S_0 \dot{M}_0 t_0 = \rho_0 V_0$, and the two flows satisfy the same continuity equation during the respective time intervals.

If one is familiar with the technique just used on the continuity equation, the conditions for similarity may be written directly by merely inspecting the equations. Appendix B gives the algebraic details for the derivation of similar conditions for the momentum and energy equations. The conditions required for similarity by the conservation equations are given in the following array:

<u>Continuity</u>

(1) $\frac{\dot{M}_0 S_0 t_0}{\rho_0 V_0} = 1$

<u>Momentum</u>

(2) $\frac{U_{w0}}{U_0} = 1$

(3) $\frac{\dot{M}_0 S_0 t_0}{M_0} = 1$

(4) $\frac{V_0 p_0 t_0}{U_0 L_0 M_0} = 1$

(5) $\frac{\tau_{w0} S_0 t_0}{U_0 M_0} = 1$

<u>Energy</u>

(6) $\frac{U_0^2}{e_0} = 1$

(7) $\frac{U_0 U_{w0}}{e_0} = 1$

(8) $\frac{U_{w0}^2}{e_0} = 1$

(9) $$\frac{e_{w0}}{e_0} = 1$$

(10) $$\frac{\dot{M}_0 S_0 t_0}{M_0} = 1$$

(11) $$\frac{p_0 V_0}{e_0 M_0} = 1$$

(12) $$\frac{\Upsilon_{w0} S_0 U_0 t_0}{M_0 e_0} = 1$$

(13) $$\frac{t_0 \dot{H}_0}{M_0 e_0} = 1$$

If the auxiliary equation $M_0 = \rho_0 V_0$ is used, conditions (1), (3), and (10) are identical. With conditions (2) and (6), conditions (7) and (8) are superfluous. Also, using condition (6), conditions (5) and (12) are identical. If conditions (11) and (6) are used in condition (4), the latter reduces to $t_0 = X_0/U_0$. The conditions required for similar solutions of the conservation equations are thus renumbered and summarized in Table I.

Table I. Conditions required for similar solutions of the conservation equations.*

| Similarity condition | Conditions imposed on two flows |
|---|---|
| (1) $\frac{\dot{M}_0 S_0 T_0}{M_0} = 1$ | The ratio of "change in mass/mass" must be identical in the two flows during the observation intervals. |
| (2) $U_{w0} = U_0$ <br> (3) $e_{w0} = e_0$ | The variables describing the incoming material must use the same scale factors as the main-flow variables. |
| (4) $t_0 = \frac{X_0}{U_0}$ | The time must scale as the ratio of "distance/velocity" scaling factors. This condition may be thought of as linking the geometric and dynamic scaling. |
| (5) $\frac{U_0^2}{e_0} = 1$ | The kinetic and internal energies must scale the same. (For two ideal gases, the Mach numbers must be identical.) |
| (6) $\frac{\Upsilon_{w0} S_0 t_0}{U_0 M_0} = 1$ | For pipe flows, the usual conditions of Reynolds-number similarity enter through the wall shear stresses. |
| (7) $\frac{p_0 V_0}{e_0 M_0} = \frac{p_0}{e_0 \rho_0} = 1$ | The ratios of "pressure/energy per unit volume" must be the same for both flows. This condition imposes a restriction on the equation of state of the gases. (For two ideal gases, the $\gamma$'s must be identical.) |
| (8) $\frac{t_0 \dot{H}_0}{e_0 M_0} = 1$ | The ratio of "heat added (or lost)/internal energy" must be identical for the two flows during the observation intervals. |

*Derived through the use of auxiliary condition $M_0 = \rho_0 V_0$.

## SECTION II:
## AUXILIARY EQUATIONS USED BY PUFL

The eight conditions given above are very general. They make no assumptions about the auxiliary equations for mass flux, energy lost (or added), equation of state, wall shear stresses, or pipe geometry. The auxiliary equations used by the PUFL calculation must now be examined in order to relate the derived similarity conditions to more specific physical requirements.

### Equation of state

First consider two ideal gases: $p = (\gamma - 1)e\rho$ and $p' = (\gamma' - 1)e'\rho'$. Then,

$$p_0 = \frac{p}{p'} = \frac{(\gamma - 1)e\rho}{(\gamma' - 1)e'\rho'} = \frac{(\gamma - 1)}{(\gamma' - 1)} e_0\rho_0.$$

Similarity condition (7) requires that $p_0/e_0\rho_0 = 1$; hence, for the two flows to be similar, it is required that $\gamma = \gamma'$.

For calculational purposes, it is usually convenient to express the equation of state as a polynomial. The following example is used here to indicate that condition (7) is really a little more general than the frequently stated similarity condition which requires that $\gamma$, the ratio of the specific heats of the two gases, be constant and equal.

Suppose that

$$p = a_1 + a_2\rho + a_3 e + a_4 e\rho + \ldots$$

and that

$$p' = \frac{p}{p_0} = \frac{a_1}{p_0} + \frac{a_2\rho}{p_0} + \frac{a_3 e}{p_0} + \frac{a_4 e\rho}{p_0} + \ldots .$$

But similarity condition (7) requires that $p_0 = e_0\rho_0 = e\rho/e'p'$. Hence,

$$p' = \frac{a_1}{e_0\rho_0} + \frac{a_2\rho'}{e_0} + \frac{a_3 e'}{\rho_0} + a_4 e'\rho' + \ldots,$$

and similarity can exist for gases with polynomial equations of state, providing that the polynomial coefficients are related as shown above; namely,

$$a_1' = \frac{a_1}{e_0\rho_0},\ a_2' = \frac{a_2}{e_0},\ a_3' = \frac{a_3}{\rho_0},\ a_4' = a_4, \ldots .$$

Note that ideal gases are a special case of the polynomial with $a_1 = a_2 = a_3 = 0$, and with $a_4 = \gamma$.

## Wall Shear Stress

The frictional stress ($\Upsilon_w$) exerted by the wall is shown by Schlichting[3] to be proportional to $\rho u^2$. The proportionality may be replaced by an equality, $\Upsilon_w = C_f(\mathrm{Re}, \epsilon/D)\,\rho u^2$, by using the dimensionless coefficient of friction ($C_f$), which is a function of the Reynolds number (Re) and the ratio of surface roughness to diameter ($\epsilon/D$). For flows at low Reynolds numbers, $C_f$ is nearly independent of surface roughness: $C_f \simeq C_f(\mathrm{Re})$. For turbulent flows at high Reynolds numbers, $C_f$ is virtually independent of the Reynolds number but depends strongly on the surface roughness: $C_f \simeq C_f(\epsilon/D)$. Because ablation requires high-energy flows that are usually well into the turbulent range, for such flows the ratio $\epsilon/D$ is of prime importance.

If $C_{f0} = C_f/C_f'$, then $\Upsilon_{w0} = C_{f0}\rho_0 U_0^2$, and similarity condition (6) may be written as

$$\frac{\Upsilon_{w0} S_0 t_0}{U_0 M_0} = \frac{C_{f0}\rho_0 U_0^2 S_0 t_0}{U_0 \rho_0 V_0} = \frac{C_{f0} U_0 S_0 t_0}{V_0} = \frac{C_{f0} X_0 S_0}{V_0} = 1.$$

Consider, for instance, two straight pipes. In this case, $S_0/V_0 = 1/R_0$. Then, condition (6) requires that $C_{f0} = R_0/X_0$. Suppose that the axial distances scale by a factor of 10 ($X_0 = 10$), and the radial distances by a factor of 20 ($R_0 = 20$). Then, similarity requires that $C_{f0} = C_f/C_f' = 20/10$ and that $C_f = 2C_f'$. Thus, the friction coefficient of the unprimed, or larger-radius, pipe must be twice as large as the primed, or smaller-radius, pipe for the flows in these two pipes to be similar.

## Ablation Equations

For dynamic gas flows, the turbulent convective heat flux may be written as $C_H \rho u h$, where $C_H$ is the dimensionless coefficient of heat transfer and h is the specific total enthalpy.[4] The rate that energy is lost by turbulent convection from the gas flow to walls having a surface area S is then $C_H \rho u h S = -\dot{H}$.

Similarity condition (8) may be written as

$$1 = \frac{t_0 \dot{H}_0}{e_0 M_0} = \frac{C_H \rho u h S}{C_H' \rho' u' h' S'} \cdot \frac{t_0}{e_0 M_0} = \frac{C_H}{C_H'} \cdot \frac{h}{h'} \cdot \frac{\rho_0 U_0 S_0 t_0}{e_0 M_0} .$$

Using similarity conditions (5) and (7), it can be shown that $h = e + p/\rho + u^2/2 = h' e_0$. Also, using condition (4) and $M_0 = \rho_0 V_0$, and introducing the expression $C_{H0} = C_H/C_H'$, similarity condition (8) may be written as

$$C_{H0} \cdot \frac{X_0 S_0}{V_0} = 1.$$

The energy required to ablate a unit mass of wall material may be written as $E_v + \eta h$, where $E_v$ is the specific energy required to vaporize the wall material and $\eta$ is a turbulent-transpiration coefficient.[4] The mass flux may then be written as

$$\dot{m} = \frac{C_H \rho u h}{E_v + \eta h}$$

where heat is transferred to the wall by turbulent convection. When this expression is used, similarity condition (1) is written as

$$\frac{\dot{M}_0 S_0 t_0}{M_0} = 1 = \left(\frac{C_H \rho u h}{E_v + \eta h}\right)\left(\frac{E_v' + \eta' h'}{C_H' \rho' u' h'}\right) \frac{S_0 t_0}{M_0} = 1.$$

For two pipes made of the same material, the assumptions $E_{v0} = E_v / E_v' = 1$ and $\eta_0 = n/n' = 1$ may not be too gross to make. Then, because $h = h'$ from conditions (5) and (7), and using condition (4), similarity condition (1) reduces to

$$C_{H0} \cdot \frac{X_0 S_0}{V_0} = 1$$

with the auxiliary conditions that $E_{v0} = 1$ and $\eta_0 = 1$. Hence, conditions (8) and (1) reduce to the same requirement.

For turbulent flows, Reynolds' analogy, which assumes that the same mechanism causes the exchange of momentum as well as of heat, is frequently used.[3] Reynolds' analogy may be expressed as $C_H = C_f/2$, and in this case as $C_{H0} = C_{f0}$. Hence, if Reynolds' analogy is assumed to hold, similarity condition (6) becomes superfluous providing that condition (1) is satisfied.

When one is trying to experimentally scale a flow during which ablated wall material adds significantly to the original mass of the flow, similarity conditions (2) and (3) become increasingly important. In discussing similarity condition (1) above, it was found that the specific heats of ablation for the two flows had to be equal $\left(E_v + \eta h = E_v' + \eta' h'\right)$, which implies the same pipe material. Hence, $e_{w0} \approx 1$ probably exists under most such conditions. Also, $e_{w0} = 1$ implies $e_0 = 1$, and condition (5) then implies that $U_0 = 1$. Thus, if significant mass addition occurs by natural ablation of wall material during a flow, a scaled flow must have the same velocity and specific internal energy (temperature) as the original flow.

If the added mass is not a significant fraction of the original mass during the flow period, then conditions (2) and (3) lose importance. In this case, only the requirement that $U_0^2 = e_0$ is important, and the possibility exists of using both conditions at a lower velocity and temperature flow as a scale model.

A number of possibilities exist for specifying the kinetic and internal energies of the ablated mass. The important requirement that the kinetic and internal energy partition for the entering material be the same in both flows must hold; i. e., $\beta = U_w^2/2e_w$ and $\beta_0 = 1$. This requirement arises from similarity conditions (2), (3), and (5).

A fairly general method presently used by PUFL to determine $e_w$ and $U_w$ is as follows. The rate at which energy enters the flow is some fraction ($\alpha$) of the rate at which energy is being lost:

$$\alpha = \frac{e_w + \frac{U_w^2}{2}}{E_v + \eta h} .$$

The fraction $\alpha$ may be determined by additional equations or may be treated as a constant for a first approximation. For $\alpha = 1$, there is no total energy loss. Using $E_{v0} = \eta_0 = 1$, $h = h'$, and conditions (2) and (3), similarity requires $\alpha_0 = e_{w0}$, which, as discussed earlier, frequently is physically required to be unity.

For the auxiliary equations used by PUFL to simulate the ablation process, the eight previously described general similarity conditions may now be restated:

(1) $\frac{C_{H0}X_0S_0}{V_0} = 1, \ E_{v0} = 1, \ \eta_0 = 1$

(2) $U_{w0} = U_0$

(3) $e_{w0} = e_0$

(2) and (3): or, with the PUFL equations, it is sufficient that $\alpha_0 = e_{w0}$, $\beta_0 = 1$

(4) $t_0 = \frac{X_0}{U_0}$

(5) $U_0^2 = e_0$

(6) $\frac{C_{f0}X_0S_0}{V_0} = 1$. If Reynolds' analogy holds, this is satisfied by condition (1).

(7) $p_0 = e_0\rho_0$. Same equation of state.

(8) Presents no new requirements if the requirements of conditions (1) are satisfied.

# SECTION III: CALCULATIONAL VERIFICATION OF THE SIMILAR CONDITIONS

In order to test the previously derived conditions for similarity, some PUFL problems are now considered. First, two finite, high-energy slugs of gases are considered to be moving into lower-energy gases in two straight-walled pipes with frictional effects, energy losses, and addition of mass. The initial conditions for an unprimed and a primed flow are shown in Fig. 1.

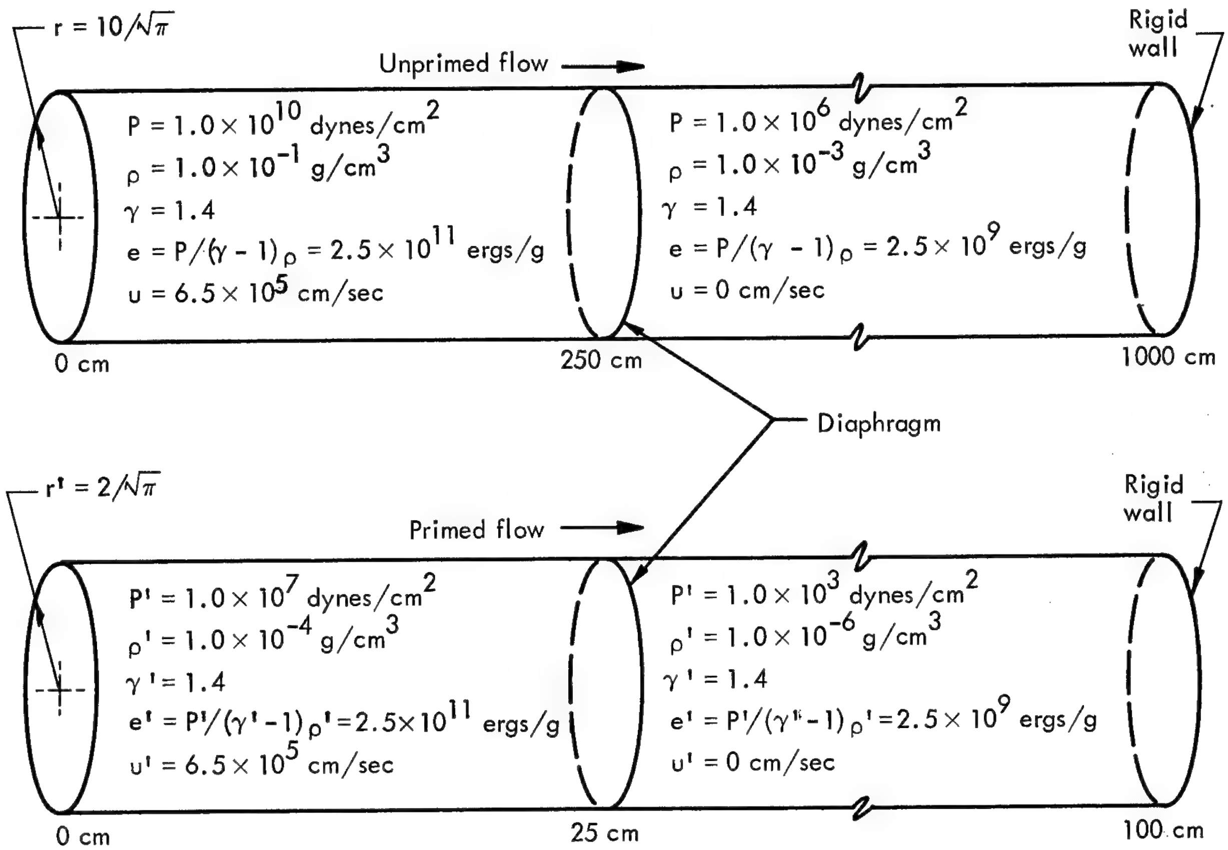



Fig. 1. Initial conditions for unprimed and primed flows.

From the initial conditions, the following scale factors are obtained: $p_0 = 1.0 \times 10^3$, $\rho_0 = 1.0 \times 10^3$, $e_0 = 1$, $U_0 = 1$, $X_0 = 10$, and $R_0 = 5$. Hence, similarity conditions (5) and (7) are satisfied. The two pipes are straight for their entire length, hence $S_0/V_0 = 1/R_0$. Condition (1) then requires that $C_{H0}\, X_0/R_0 = 1$ or that $C_{H0} = R_0/X_0 = 1/2$, hence $C_H = C_H'/2$. For the two problems considered here, $C_H = 0.002$ and $C_H' = 0.004$ have been picked. In addition, $E_V = E_V' = 4.0 \times 10^{10}$ and $\eta = \eta' = 0.1$ have been chosen. It is assumed that one-half of the energy lost from the flow is returned as ablated mass ($\alpha = \alpha' = 0.5$), and that $\beta = \beta' = 0.1 = U_w^2/2e_w$ to

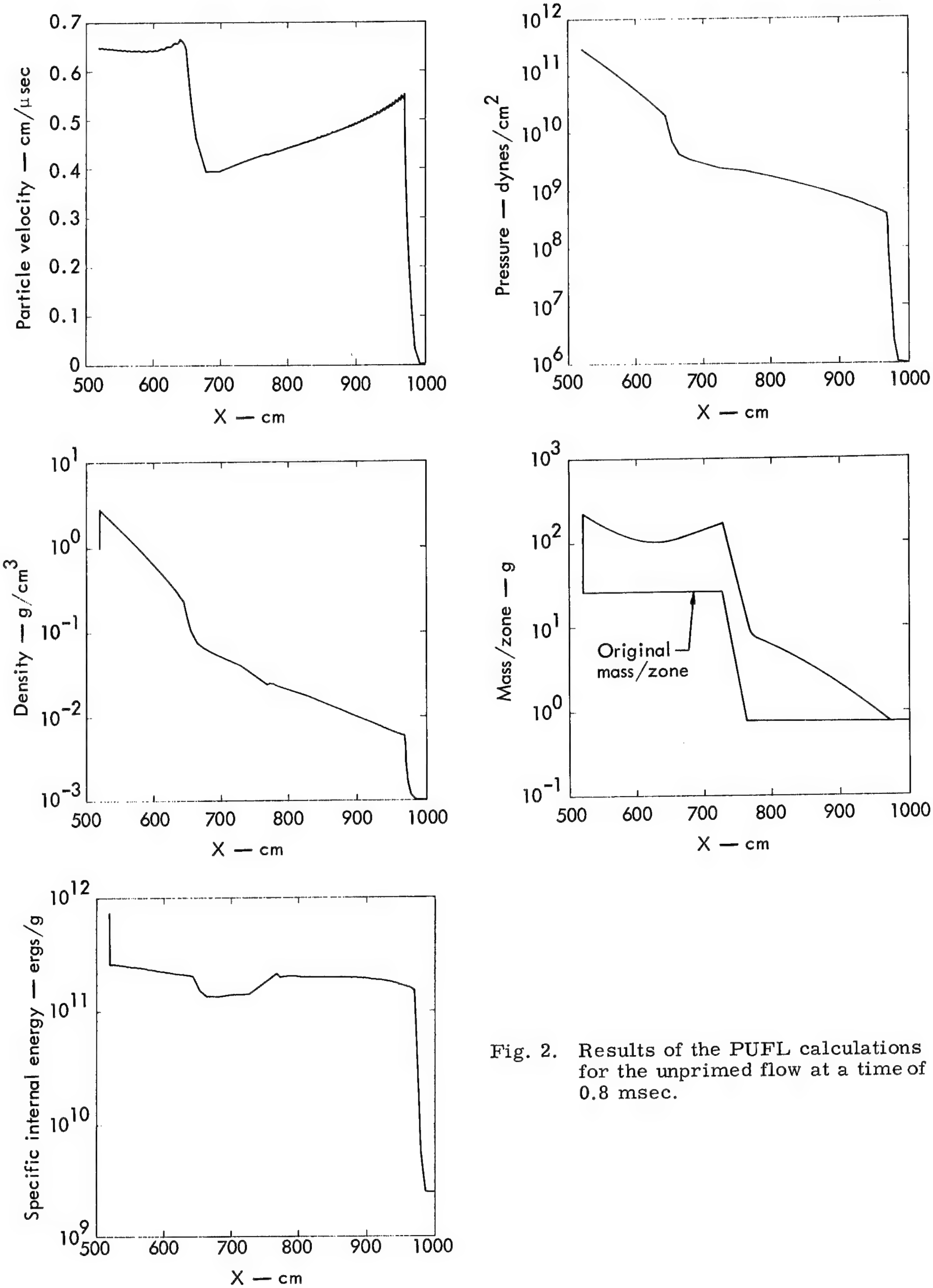
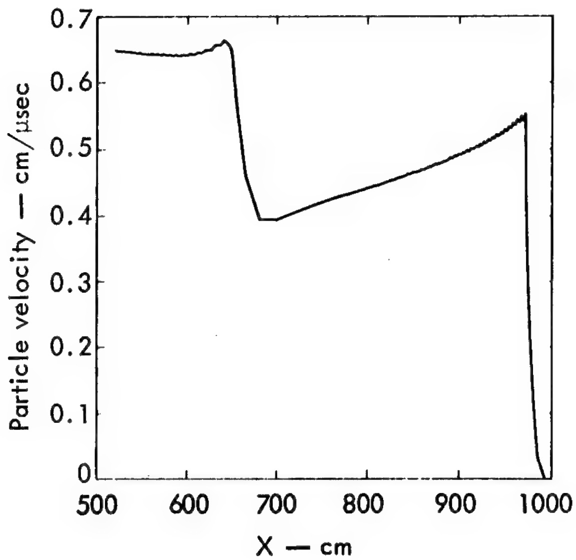
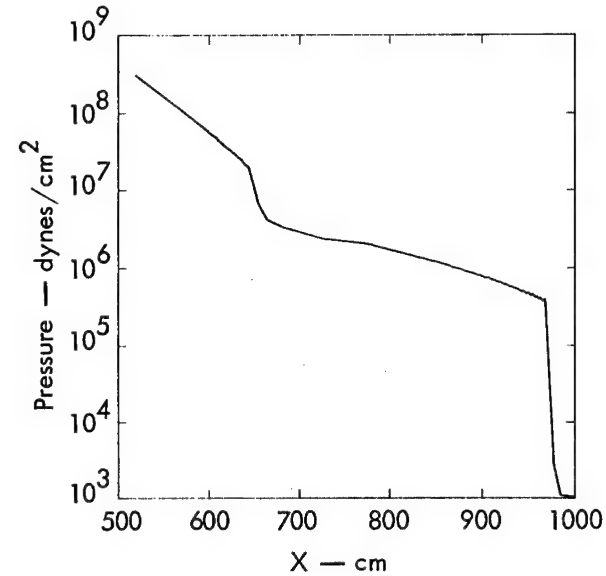
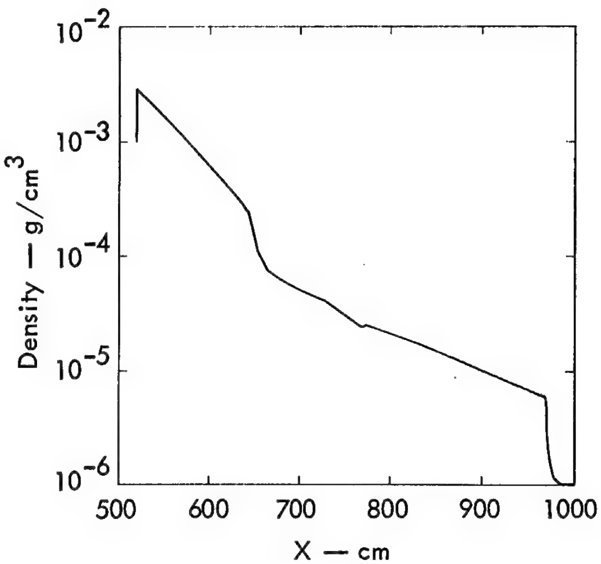



Fig. 2. Results of the PUFL calculations for the unprimed flow at a time of 0.8 msec.



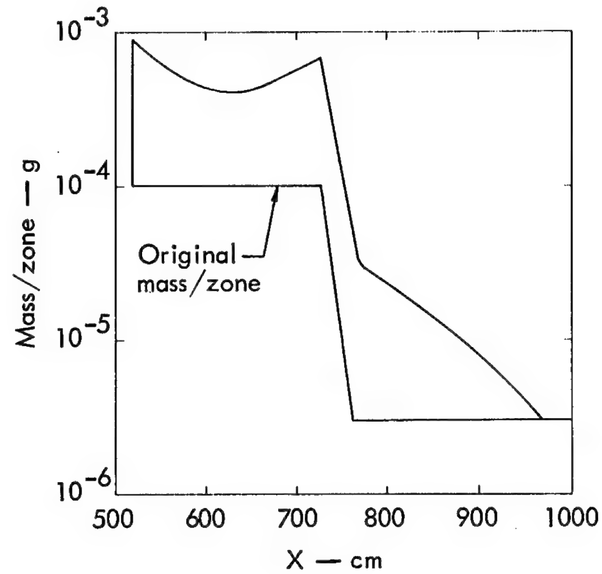
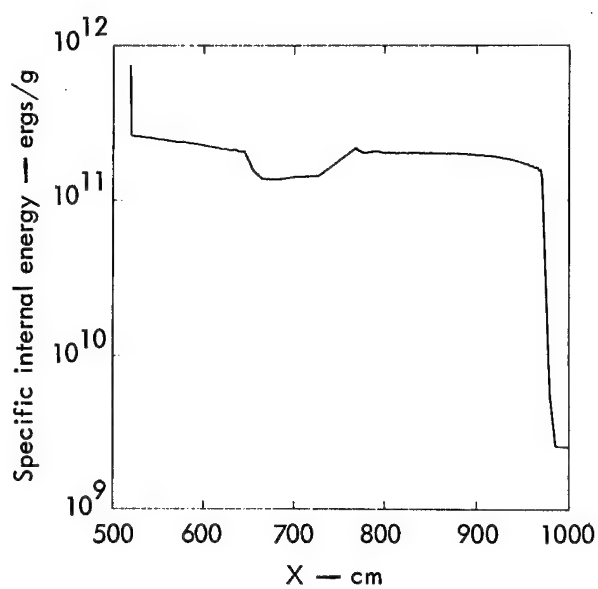




Fig. 3. Results of the PUFL calculations for the primed flow at a time of 0.08 msec.



satisfy conditions (2) and (3). Condition (4) is satisfied if the time is scaled by the factor $10 = t_0 = X_0/U_0$. Reynolds' analogy is assumed to hold (i.e., $C_{H0} = C_{f0}$), and condition (6) is satisfied by $C_f = 2C_H = 0.004$ and $C_f' = 2C_H' = 0.008$.

The results of the PUFL calculations for these unprimed and primed flows are shown at times of 0.8 msec and 0.08 msec in Figs. 2 and 3 respectively. These figures show that the solutions are similar.

The next two problems deal with pipes having non-constant radii and converging nozzles. The surface area and volume for the frustums of cones (i.e., cylindrical nozzles) are

$$S = \pi(r_1 + r_2)\sqrt{(\Delta x)^2 + (r_1 - r_2)^2}$$

and

$$V = \frac{\pi\Delta x}{3}\left(r_1^2 + r_1 r_2 + r_2^2\right).$$

Hence, in order to maintain a constant ratio of $S_0/V_0 = 1/R_0$ for nozzle problems, the axial length and radii must scale the same (i.e., $X_0 = R_0$).

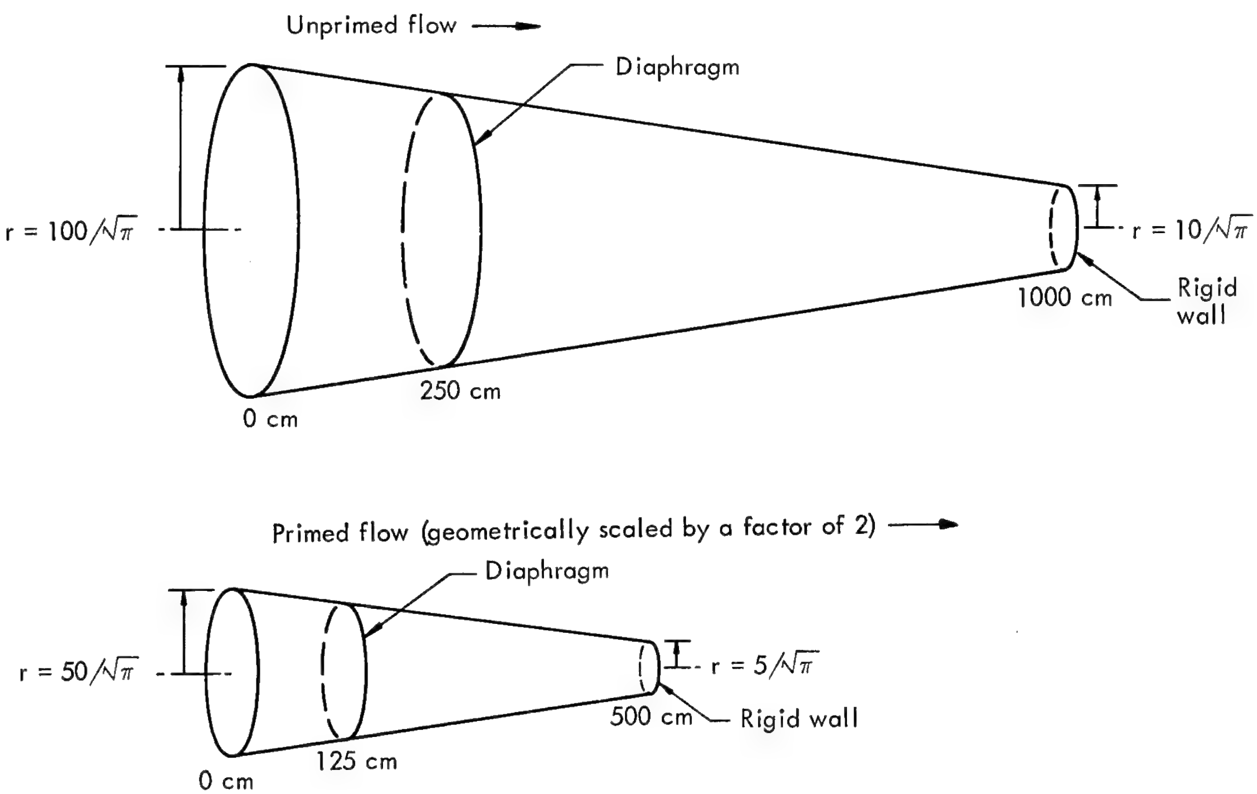


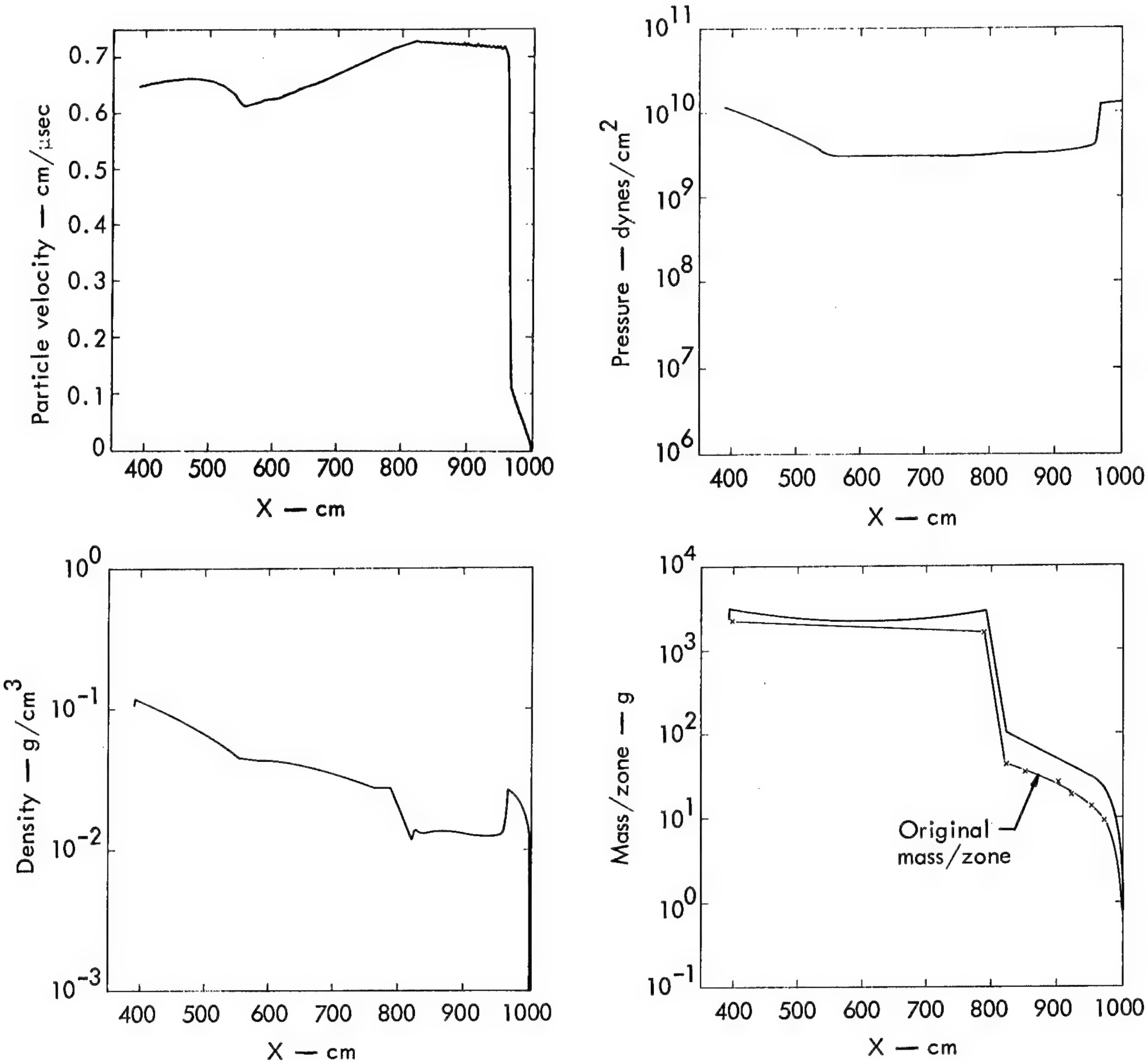

Fig. 4. Initial conditions for unprimed and primed nozzle flows (flow parameters are the same as those in Fig. 1).

The two flows considered next are again finite, high-energy slugs of gases moving into lower-energy gases, just as in the last problem. The geometry of the unprimed and primed flows are shown in Fig. 4. This geometry imposes the scale factors $R_0 = X_0 = 2$. From the initial conditions, $U_0 = 1$; hence, $t_0 = X_0/U_0 = 2 = t/t'$. The pipe walls are allowed to radially expand a distance equal to their original radius. The expansion time is $5 \times 10^{-4}$ sec for the unprimed flow and $2.5 \times 10^{-4}$ sec for the primed flow. Under these conditions, at any instant the surface-to-volume ratios of the expanding nozzles remain constant: $S_0/V_0 = 1/R_0 = 1/2$. These problems use $C_f = 0.008$, $C_f' = 0.008$, $C_H = 0.004$, $C_H' = 0.004$, along with the same ablation conditions as the previous problems. Hence, the remaining conditions for similarity are satisfied.

The results of the PUFL calculations for these two flows are shown in Fig. 5 (p.16) for the unprimed flow at a time of 0.6 msec and in Fig. 6 (p.17) for the primed flow at a time of 0.3 msec. These results are shown right after the shock starts to reflect from the rigid right-hand wall. The results are seen to be similar, indicating the validity of the derived conditions.

## SUMMARY

For turbulent flows, the familiar Reynolds-number requirement for similarity may be reduced to requirements on surface roughness and surface-to-volume ratios. For heat sources, similarity requires that during the scaled observation intervals, the ratio of "heat added/internal energy" be identical. Similarity for mass sources requires that the ratio of "change in mass/mass" be identical in two flows during the scaled observation intervals. Also, variables describing entering mass must use the same scale factors as the main-flow variables. For gas flows where ablation is not important, equal Mach numbers are required for similarity. However, if significant mass entrainment occurs during the observed time intervals of the two flows, and if the entrained mass has identical properties when it enters the two flows, then similarity requires that the two observed main flows have the same velocity and temperature. Geometrically similar turbulent flows, in which heat transfer and friction are related by Reynolds' analogy and in which equivalent assumptions are made about the entering mass, are shown to readily satisfy the derived similarity conditions.



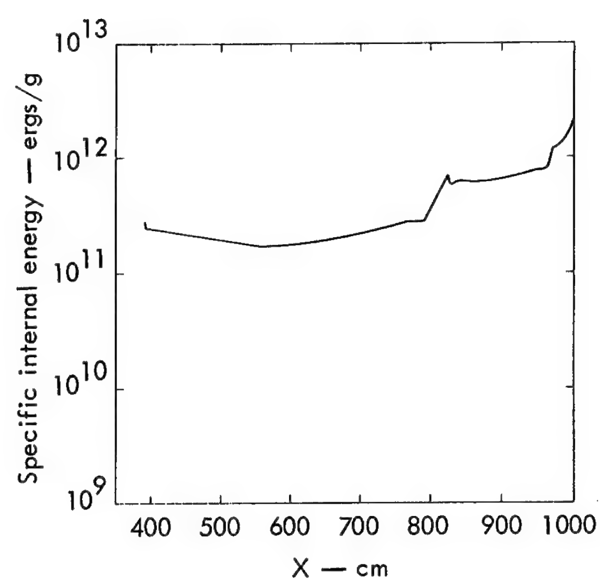
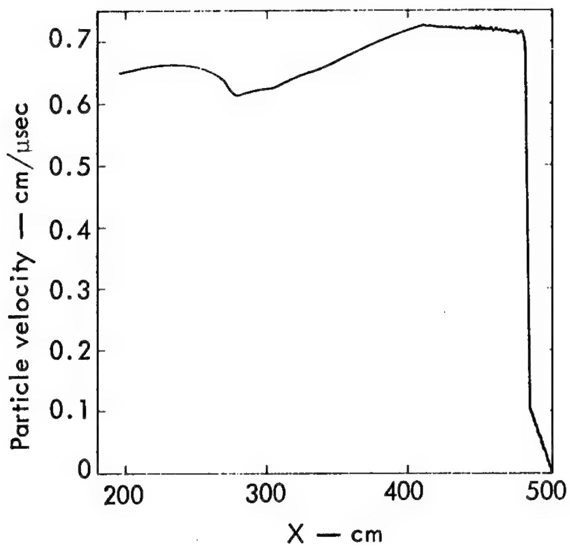
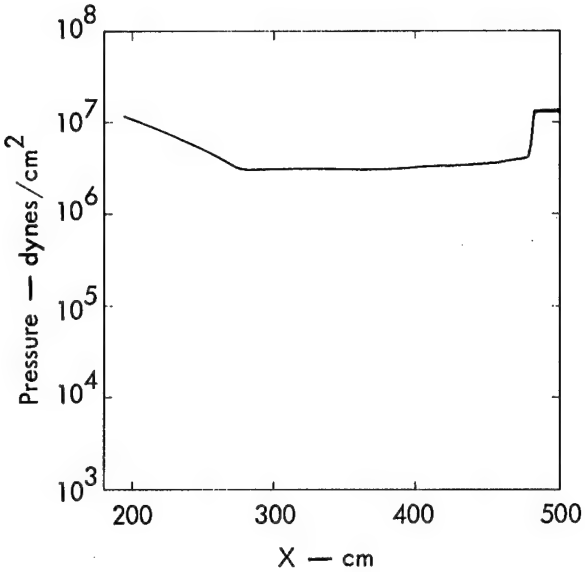
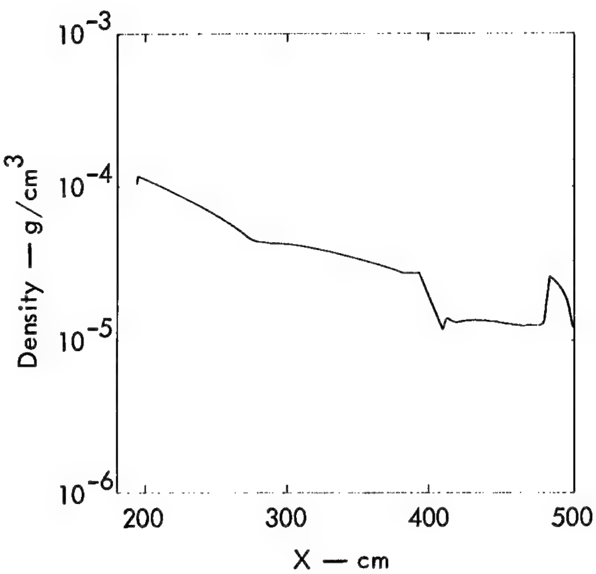




Fig. 5. Results of the PUFL calculations for the unprimed nozzle flow at a time of 0.6 msec.



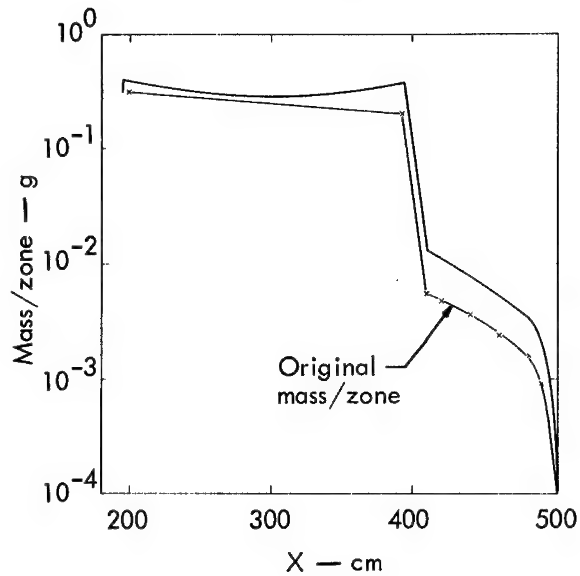
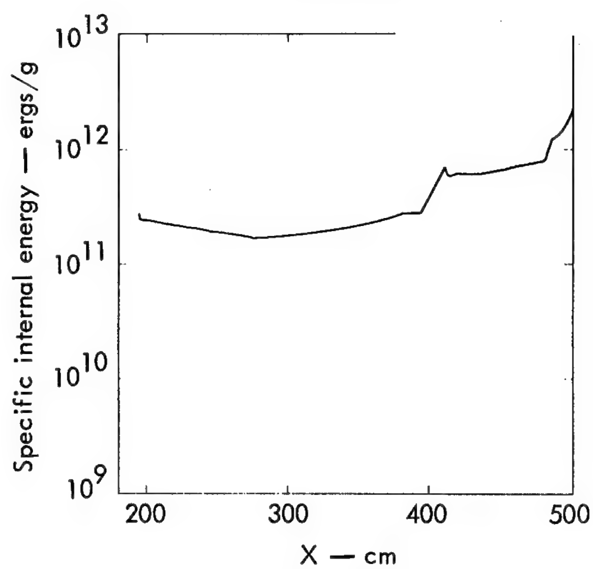




Fig. 6. Results of the PUFL calculations for the primed nozzle flow at a time of 0.3 msec.



ACKNOWLEDGMENTS

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REFERENCES

[1]B. K. Crowley, PUFL, an "Almost-Lagrangian" Gas Dynamic Calculation for Pipe Flows with Mass Entrainment, UCRL-14536, Lawrence Radiation Laboratory, Livermore (1966).

[2]G. Birkhoff, Hydrodynamics: A Study in Logic, Fact and Similitude (Princeton University Press, Princeton, New Jersey, 1960).

[3]H. Schlichting, Boundary Layer Theory (McGraw-Hill Book Co., New York, 1960).

[4]P. H. Rose and E. Offenhartz, Ablation Measurements in Turbulent Flow, AVCO-PR-114, Avco-Everett Research Laboratory (1959).

# APPENDIX A

The approach to similarity used in this report considers two arbitrary physical flows, the primed and the unprimed. At some instant in time, the two flows are examined and ratios of all of the pertinent variables are obtained. The requirements imposed on these ratios are then determined. These requirements must be satisfied in order for the differential conservation equations to be identical in both flows.

This approach differs slightly from the approach traditionally used. It has been chosen, however, because of its slightly closer relationship to the frequently encountered physical situation of having two experiments and asking, "What relationships must exist between the parameters in the two experiments at this instant in order that one may represent a scaled model of the other as time increases?" It is shown in the following discussion that the approach used in this report and the one traditionally encountered in similarity analysis give equivalent results.

For the more traditional approach, dimensionless quantities are obtained from the flow parameters by selecting certain suitable and characteristic magnitudes for the flow being considered. For example, consider a pipe flow described by a set of unprimed variables. A characteristic length of the flow ($X_R$) is set equal to the radius, and a reference time ($t_R$) is frequently obtained from a reference velocity ($U_R$) as $t_R = X_R/U_R$. A reference density ($\rho_R$), mass flux ($\dot{M}_R$), surface area ($S_R$), and volume ($V_R$) are also selected. A set of dimensionless variables, denoted by asterisks, is then obtained from the physical flow variables and from the reference magnitudes:

$$X^* = \frac{X}{X_R},\ r^* = \frac{r}{r_R},\ V^* = \frac{V}{V_R},\ S^* = \frac{S}{S_R},\ \dot{m}^* = \frac{\dot{m}}{\dot{m}_R},\ t^* = \frac{t}{t_R},\ \text{and } \rho^* = \frac{\rho}{\rho_R}\ .$$

Derivatives in the dimensionless system are related to the derivatives for the flow variables as follows:

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial X^*} \cdot \frac{\partial X^*}{\partial X} = \frac{1}{X_R} \cdot \frac{\partial}{\partial X^*}$$

and

$$\frac{D}{Dt} = \frac{D}{Dt^*} \cdot \frac{Dt^*}{Dt} = \frac{1}{t_R} \cdot \frac{D}{Dt^*}\ .$$

The continuity equation for the flow variables is

$$\frac{D\rho}{Dt} = -\frac{\rho}{V} \cdot \frac{dV}{dt} + \frac{S\dot{m}}{V}.$$

Using the relationships defined above, the continuity equation becomes

$$\frac{D\rho^*\rho_R}{t_R\, Dt^*} = -\frac{\rho^*\rho_R dV^*}{V^* V_R t_R dt^*} + \frac{S^* S_R \dot{m}^* \dot{m}_R}{V^* V_R}.$$

Hence,

$$\frac{D\rho^*}{Dt^*} = -\frac{\rho^*}{V^*} \cdot \frac{dV^*}{dt^*} + \frac{S_R \dot{m}_R t_R}{\rho_R V_R} \cdot \frac{S^* \dot{m}^*}{V^*}$$

is the dimensionless form of the continuity equation for the flow.

Now consider another flow denoted by a set of primed variables. A dimensionless set of variables are obtained from the primed flow variables by using a primed set of reference values. Following the same procedure as that used for the unprimed flow, the dimensionless continuity equation for the primed flow is

$$\frac{D\rho'^*}{Dt'^*} = -\frac{\rho'^*}{V'^*} \frac{dV'^*}{dt'^*} + \frac{S'_R \dot{m}'_R t'_R}{\rho'_R V'_R} \frac{S'^* \dot{m}'^*}{V'^*}.$$

In order for the primed and unprimed flows to be dynamically similar, the primed and unprimed dimensionless continuity equations must be identical. Hence, dynamic similarity requires that

$$\frac{S_R \dot{m}_R t_R}{\rho_R V_R} = \frac{S'_R \dot{m}_R t'_R}{\rho'_R V'_R}.$$

The reference values may be chosen arbitrarily. Hence, no loss of generality results if the reference values are chosen so that all of the primed and unprimed dimensionless variables are equal to unity. Then, for instance,

$$S^* = \frac{S}{S_R} = S'^* = \frac{S'}{S'_R} = 1$$

and

$$\frac{S}{S'} = \frac{S_R}{S'_R}.$$

Hence, the ratio of the variables in the two flows can be expressed in terms of the ratio of the reference values. This report gives the ratio of reference values as $S_R/S'_R = S_0 = S/S'$. Thus, the notation used in the report may be expressed as

$$S_0 = \frac{S_R}{S_R'} = \frac{S}{S'},\ V_0 = \frac{V_R}{V_R'} = \frac{V}{V'},\ t_0 = \frac{t_R}{t_R'} = \frac{t}{t'},\ \rho_0 = \frac{\rho_R}{\rho_R'} = \frac{\rho}{\rho'},\ \text{and}\ \dot{M} = \frac{\dot{m}_R}{\dot{m}_R'} = \frac{\dot{m}}{\dot{m}'}.$$

The previously derived condition for dynamic similarity of the continuity equation may now be written as

$$\frac{S_R \dot{m}_R t_R}{S_R' \dot{m}_R' t_R'} = \frac{\rho_R V_R}{\rho_R' V_R'} = S_0 \dot{M} t_0 = \rho_0 V_0$$

or as

$$\frac{S_0 \dot{M} t_0}{\rho_0 V_0} = 1,$$

which is the same as that obtained in the report.

Generally, the traditional approach of determining whether two flows are similar involves comparing groups of reference values like $S_R \dot{m}_R t_R / \rho_R V_R$ that are derived from continuity. When all such groups (from the complete set of equations) are numerically equal for two flows, the flows are called similar. The flows then behave in a similar manner as time advances.

Although the reference values have dimensions, the referred-to groups of these reference values are dimensionless. Their nondimensional character is due to the fact that they are coefficients of terms in the dimensionless set of equations. For instance, the Reynolds number that is frequently used in similarity work is just such a dimensionless group obtained from certain forms of the momentum equation.

For the PUFL equations, it is not intuitively obvious what values should be chosen for reference or as characteristic values for some variables (e.g., the mass flux). Hence, it is legitimate to examine both flows at some instant and to choose the reference values so that the dimensionless variables have, at that instant, a numerical value of unity. Then the ratio of reference values becomes simply the ratio of flow variables. This is the essence of what is done in this report.

# APPENDIX B

The momentum and energy equations given in this report are examined in the following discussion, and conditions for similar solutions to the equations are derived.

## Momentum

The momentum equation is

$$\frac{Du}{Dt} = \frac{1}{m}\left[\dot{m}S(u_w - u) - V\frac{\partial p}{\partial x} - \tau_w S\right].$$

Substituting for the primed variables gives

$$\frac{U_0}{t_0}\cdot\frac{Du'}{Dt'} = \frac{1}{m'M_0}\left[\dot{m}'\dot{M}_0 S' S_0(U_{w0}u_w' - U_0 u') - \frac{V_0 V' p_0}{m' M_0 X_0}\cdot\frac{\partial p'}{\partial x'} - \frac{\tau_{w0}\tau_w' S_0 S'}{m' M_0}\right].$$

The energy equation in the primed system thus is

$$\frac{Du'}{Dt} = \frac{t_0}{U_0}\cdot\frac{\dot{M}_0 S_0 U_{w0}}{M_0}\cdot\frac{\dot{m}'S'}{m'}u_w' - \frac{t_0\dot{M}_0 S_0 U_0}{U_0 M_0}\cdot\frac{\dot{m}'S'}{m'}u' - \frac{t_0}{U_0}\cdot\frac{V_0 p_0}{X_0 M_0}\cdot\frac{V'}{m'}\cdot\frac{\partial p'}{\partial x'}$$

$$- \frac{t_0}{U_0}\cdot\frac{\tau_{w0}S_0}{M_0}\cdot\frac{\tau_w' S'}{m'}.$$

From the above, the conditions required for similar solutions are:

(1) $$\frac{t_0\dot{M}_0 S_0 U_{w0}}{M_0 U_0} = 1$$

(2) $$\frac{t_0\dot{M}_0 S_0}{M_0} = 1$$

(3) $$\frac{t_0 V_0 p_0}{U_0 X_0 M_0} = 1$$

(4) $$\frac{t_0\tau_{w0}S_0}{U_0 M_0} = 1$$

Since condition (2) requires that $\dot{M}_0 S_0 t_0 / M_0 = 1$, condition (1) imposes the additional requirement that $U_{w0}/U_0 = 1$.

## Energy

The energy equation is

$$\frac{De}{Dt} = \frac{\dot{m}S}{m} \cdot \frac{(u - u_w)^2}{2} + \frac{\dot{m}S}{m}(e_w - e) - \frac{p}{m} \cdot \frac{DV}{Dt} + \frac{\Upsilon_w S|u|}{m} + \frac{\dot{H}}{m} .$$

Substituting for the primed variables gives

$$\frac{e_0}{t_0} \cdot \frac{De'}{Dt'} = \frac{\dot{M}_0 S_0}{M_0} \cdot \frac{\dot{m}'S'}{m'} \cdot \frac{(u'U_0 - u_w'U_{w0})^2}{2} + \frac{\dot{M}_0 \dot{m}'S'S_0}{m'M_0}(e_w e_{w0}' - e'e_0)$$

$$- \frac{p_0 p'}{M_0 m'} \cdot \frac{V_0}{t_0} \cdot \frac{dV'}{dt'} + \frac{\Upsilon_{w0}\Upsilon_{w'} S_0 S'|u'|}{m'M_0} U_0 + \frac{\dot{H}'\dot{H}_0}{m'M_0} .$$

The energy equation in the primed system thus is

$$\frac{De'}{Dt'} = \frac{t_0}{e_0} \cdot \frac{\dot{M}_0 S_0}{M_0} U_0^2 \frac{\dot{m}'S'}{m'} \cdot \frac{u'^2}{2} - \frac{t_0}{e_0} \cdot \frac{\dot{M}_0 S_0}{M_0} U_{w0} U_0 \frac{\dot{m}'S'u'u_w'}{m'}$$

$$+ \frac{t_0}{e_0} \cdot \frac{\dot{M}_0 S_0}{M_0} U_0^2 \frac{\dot{m}'S'}{m'} \cdot \frac{u_w'^2}{2} + \frac{t_0 \dot{M}_0 S_0}{e_0 M_0} e_{w0} \frac{\dot{m}'S'u'u_w'}{m'}$$

$$- \frac{t_0 \dot{M}_0 S_0 e_0}{e_0 M_0} \cdot \frac{\dot{m}'S'e'}{m'} - \frac{t_0 p_0 V_0}{e_0 M_0 t_0} \cdot \frac{\rho' DV'}{m' Dt'}$$

$$+ \frac{t_0 \Upsilon_{w0} S_0 U_0}{e_0 M_0} \cdot \frac{\Upsilon_w' S'|u'|}{m'} - \frac{t_0 \dot{H}_0 \dot{H}'}{e_0 M_0 m'} .$$

The conditions for similar solutions imposed by the conservation of energy thus are:

(1) $\dfrac{t_0 \dot{M}_0 S_0 U_0^2}{e_0 M_0} = 1$

(2) $\dfrac{t_0 \dot{M}_0 S_0 U_0 U_{w0}}{e_0 M_0} = 1$

(3) $\dfrac{t_0 \dot{M}_0 S_0 U_{w0}^2}{e_0 M_0} = 1$

(4) $\dfrac{t_0 \dot{M}_0 S_0 e_{w0}}{e_0 M_0} = 1$

(5) $$\frac{t_0 \dot{M}_0 S_0}{M_0} = 1$$

(6) $$\frac{p_0 V_0}{e_0 M_0} = 1$$

(7) $$\frac{t_0 \tau_{w0} S_0 U_0}{e_0 M_0} = 1$$

(8) $$\frac{t_0 \dot{H}_0}{e_0 M_0} = 1$$

Since condition (5) requires that $\dot{M}_0 S_0 t_0 = M_0$, conditions (1) through (4) may be simplified to:

(1) $$\frac{U_0^2}{e_0} = 1$$

(2) $$\frac{U_0 U_{w0}}{e_0} = 1$$

(3) $$\frac{U_{w0}^2}{e_0} = 1$$

(4) $$\frac{e_{w0}}{e_0} = 1$$

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